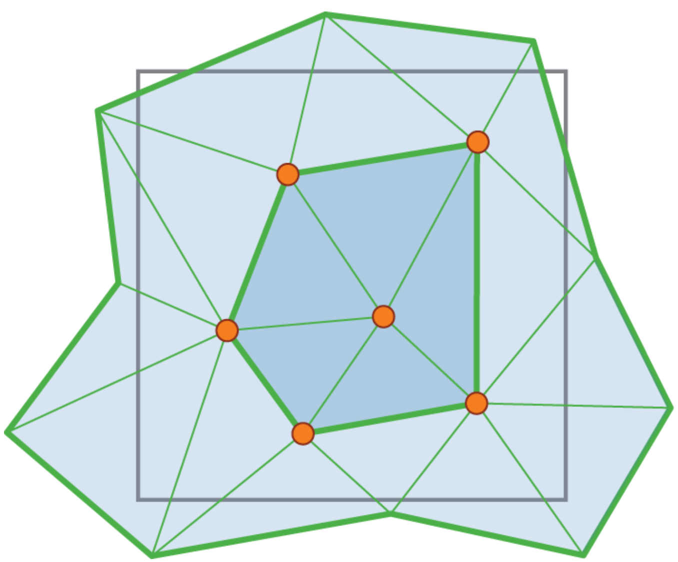
The PR-star Octree:

A *spatio-topological* data structure for tetrahedral meshes



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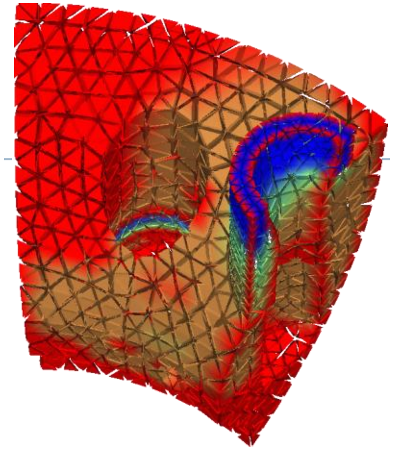
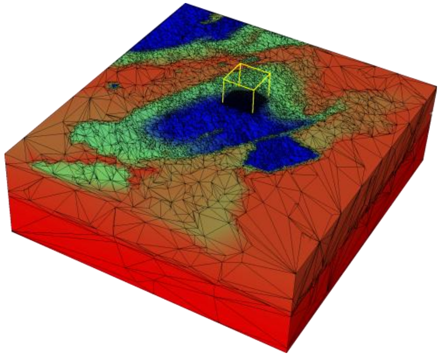
Motivation





- Tetrahedral meshes
  - Increasingly important for analysis and visualization of scientific datasets
  - Captured/simulated at increasingly fine resolution
- Mesh connectivity
  - Important for many tasks that process the mesh
    - Navigation, visibility, morphology, discrete curvature estimates ray tracing/path following, simplification and repair, etc…
  - Expensive to encode
  - Representations typically are catered to needs of application
- Processing rates (CPU/GPU) increasing faster than memory
  - Favor reductions in memory over those in computing

PR-star Octree

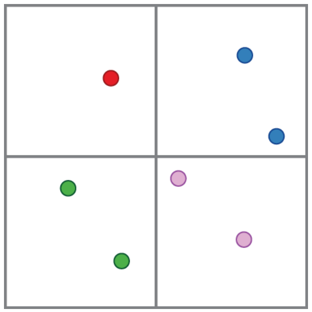
Contributions

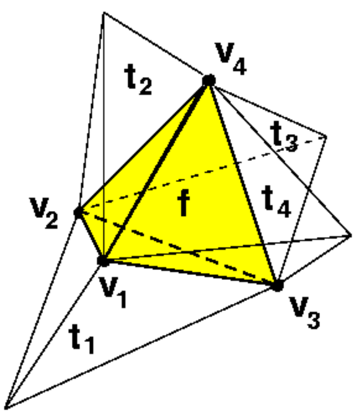
- "Topology through space"
  - Topological connectivity queries
    through spatial index on embedding space
- Encode just enough information to enable efficient reconstruction of all topological relations
  - Allows optimal application-dependent
    local data structures to be generated at runtime
  - Construction costs amortized over multiple coherent queries
- Streaming algorithms over dataset
  - Boundary determination, local curvature estimates, simplification
  - Many more…
- Benefits of this representation increase with dataset size

Related Work

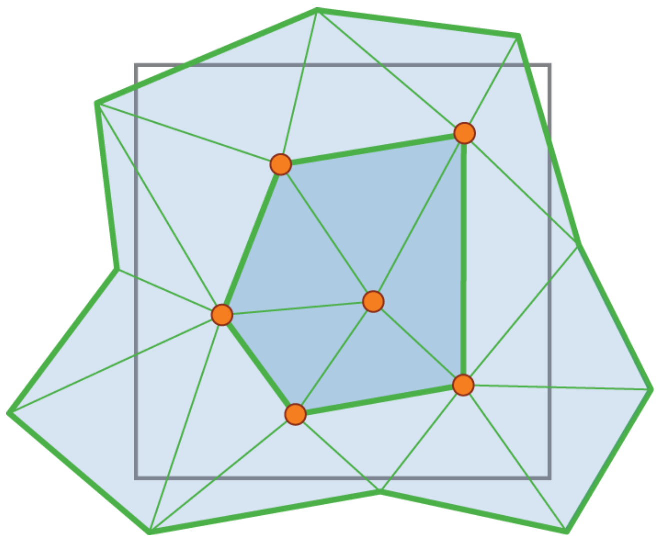
- Spatial data structures
  - Focus is on efficient spatial queries
    - e.g. point location, (*k*)- nearest neighbor query
  - Points:
    - PR- quadtrees, octrees and kd-trees [Samet:2006]
  - Polygons, edges and graphs; Triangles:
    - PM-family of quadtrees – PM1-, PM2-, PM3-, PMR-
  - Tetrahedral meshes [De Floriani *et al.*:2010]
- Topological data structures
  - Focus is on efficient connectivity queries
  - Incidence-based – IG [Edelsbrunner:1987]
  - Adjacency-based – IA [Paoluzzi:1993; Nielson:1997]
- Spatial index on triangle mesh for out-of-core processing [Cignoni:2003] or for expensive processing [Dey et al.: 2010]

Talk overview

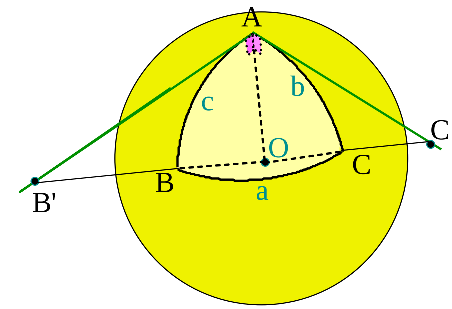


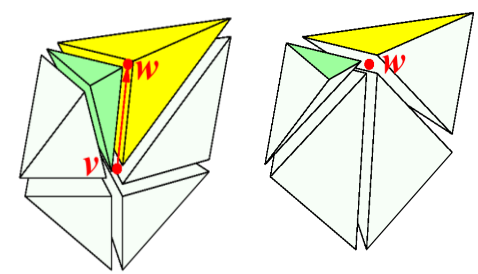


Background



PR-star Octree

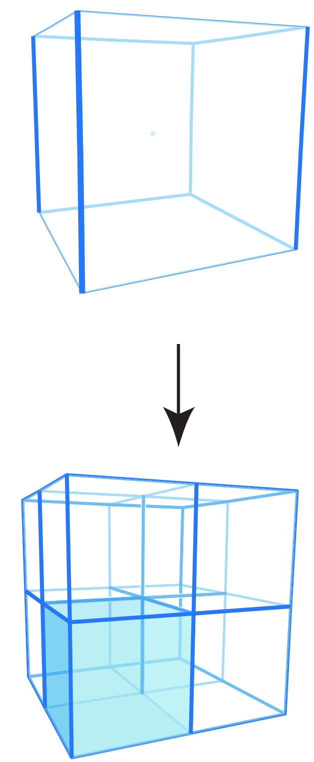


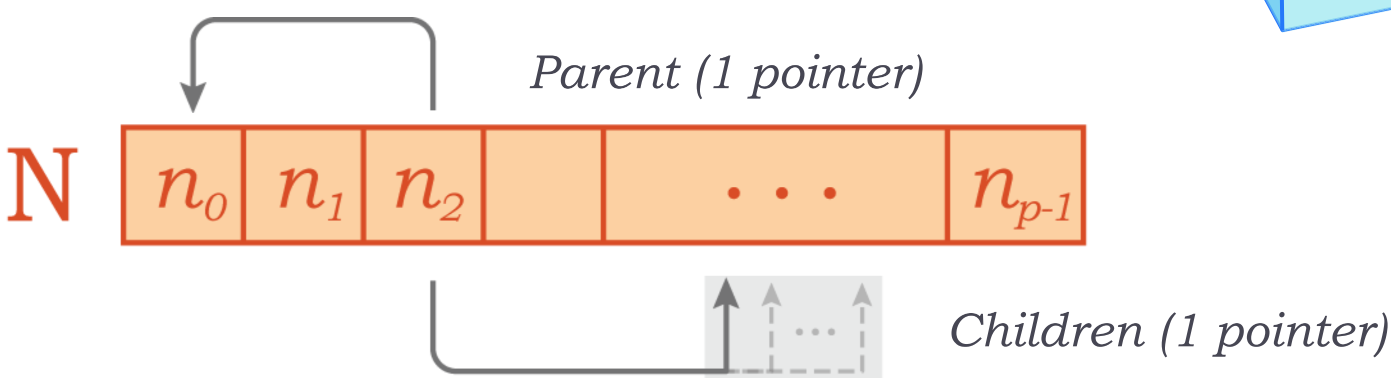


Applications

Region Octrees

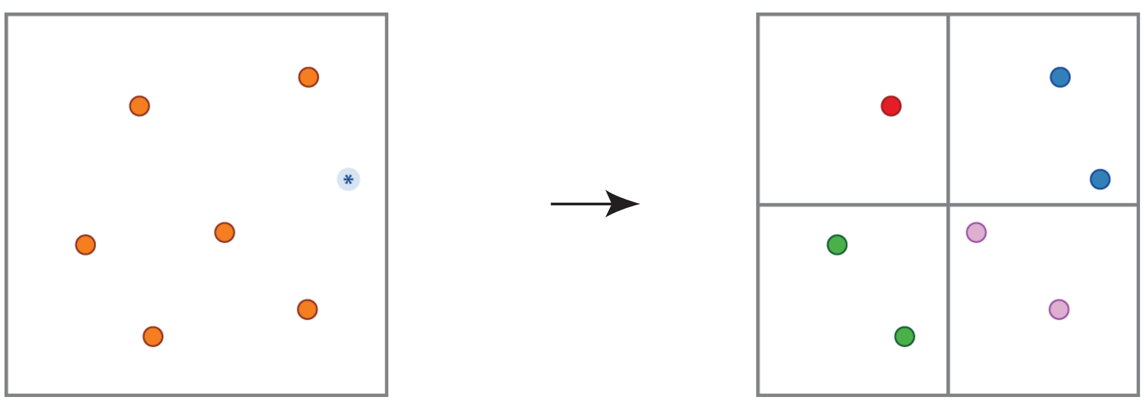
- Hierarchical domain decomposition
- Regular refinement
  - Each cubic parent node is replaced by eight children nodes covering its domain
- Root node
  - Cubic node covering entire domain
- Leaf node
  - Cubic node without children
  - Non-leaf nodes are called *internal nodes*





PR Octree:
Point Region Octree

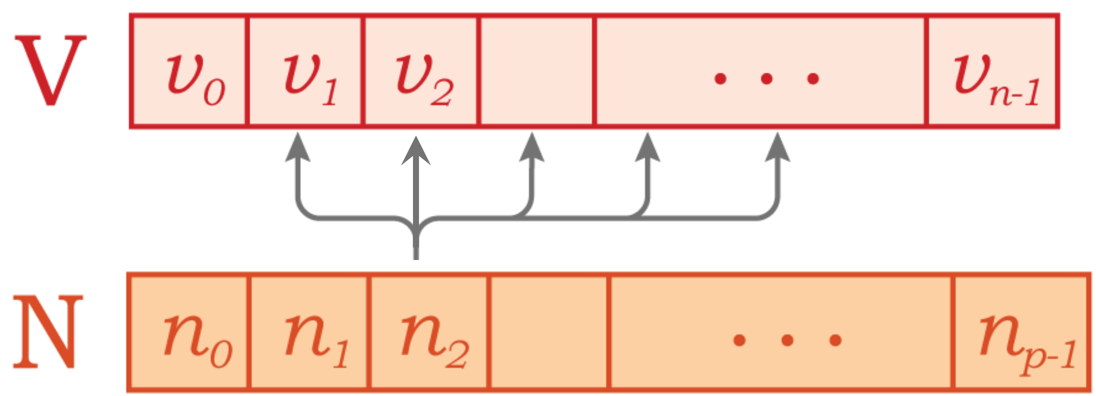
- Region octree used as spatial index on a set of points
  - Points are uniquely indexed by a single leaf node
- Bucket threshold k_v
  - Used to decide when to split a node
    - Decomposition entirely dependent on k_v
- A node is considered *full* when it indexes k_v points
  - Redistribute points to children upon insertion into *full* leaf node



$k_v = 6$

PR Octree:
Representation

- An array of points in R^3 – ***V***
- A set (array) of octree nodes – ***N***
  - Each leaf node ***n*** in ***N*** indexes the set of at most k_v points from ***V*** that lie within its domain

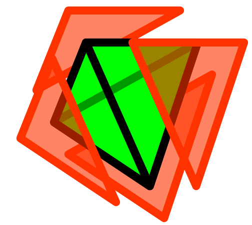


Topological Connectivity Relations

- Fundamental *connectivity* primitives for mesh elements

Boundary relations – $R_{p,q}$ $(p<q)$

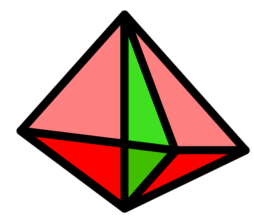
- Set of q-simplices that are a face of a given p-simplex
- e.g. $R_{3,0}$ is the *Tetrahedron-Vertex* relation



$R_{3,2}$

Co-boundary relations – $R_{q,p}$ $(p<q)$

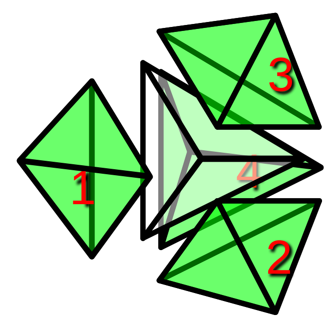
- Set of simplices that have a given simplex as a face
- e.g. $R_{0,3}$ is the *Vertex-Tetrahedron* relation
  - The tetrahedra in the *star* of v



$R_{2,3}$

Adjacency relations – $R_{p,p}$

- Set of p-simplices that adjacent to a given simplex along a p-1 face $(p>0)$ or an edge $(p=0)$
- e.g. $R_{3,3}$ is the *Tetrahedron-Tetrahedron* relation



$R_{3,3}$

Topological Data Structures

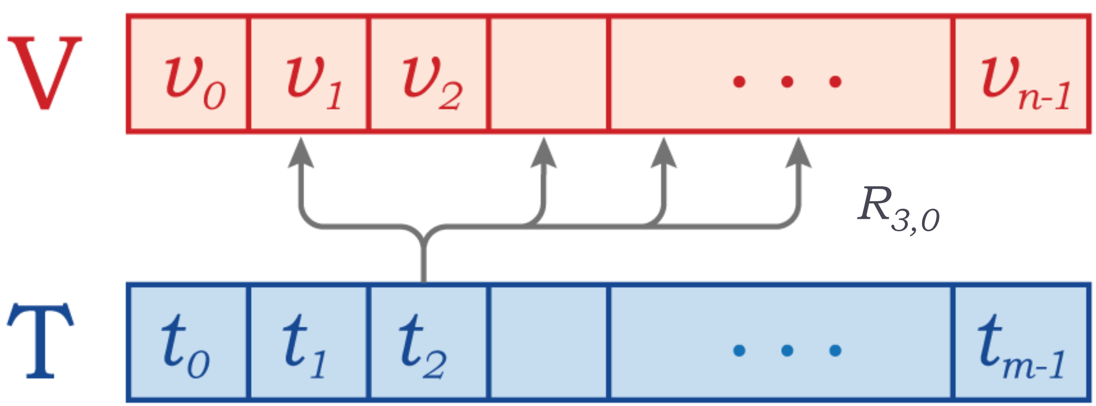
- Explicitly encode a subset of the topological relations
- Implicitly encode a (larger) subset of the relations
  - Reconstruct relevant neighborhoods from encoded relations at runtime

- Application-dependent data formulations
  - *Incidence-based* data structures
    - e.g. Incidence Graph [Edelsbrunner:1987]
  - *Adjacency-based* data structures
    - e.g. Indexed data structure with Adjacency (IA) [Paoluzzi *et al*:1993]

- Adjacency-based data structures more compact when we are mainly interested in *top cells* [DeFloriani and Hui : 2006]

Indexed tetrahedral mesh

- Array of vertices **V**
  - Each vertex v_i encodes a position (x,y,z) and possibly other attributes
- Array of tetrahedra **T**
  - Each tetrahedron t_j encodes the index in **V** of its vertices and possibly other attributes

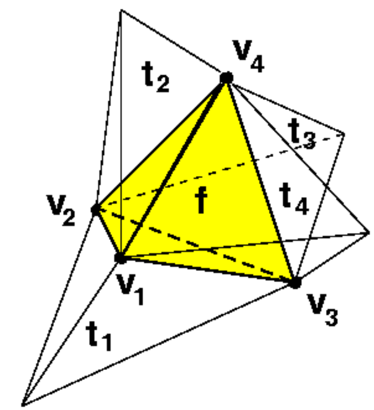


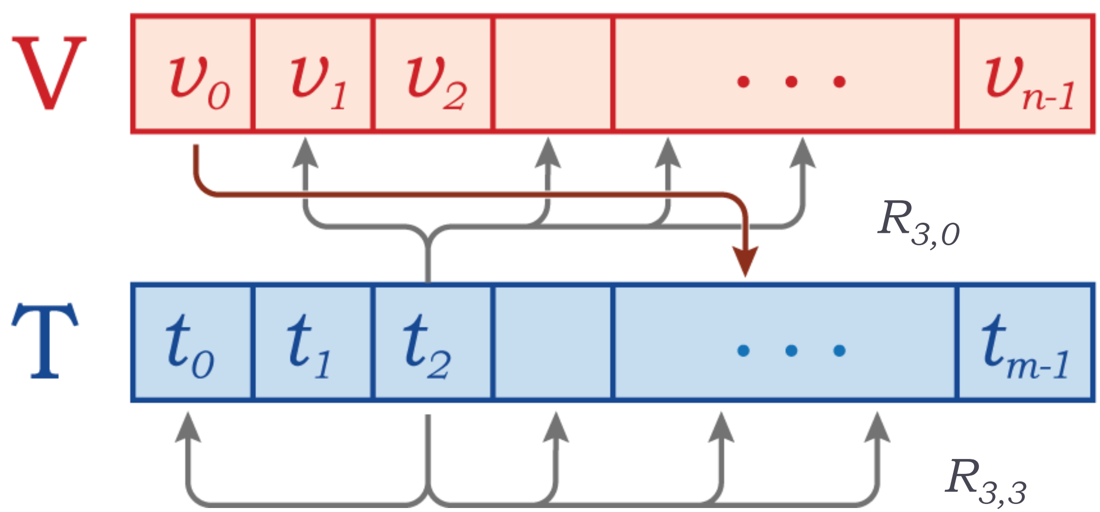
$$v_i = \{x, y, z\}$$

$$t_i = \{i_{v0}, i_{v1}, i_{v2}, i_{v3}\}$$

IA data structure:
Indexed tetrahedral mesh with Adjacencies

- Array of vertices **V**
  - Encodes position of each vertex
  - Encodes a single incident tetrahedron in **T**
- Array of tetrahedra **T**
  - Encodes indices of four vertices in **V**
  - Encodes indices of four adjacent tetrahedra in **T**



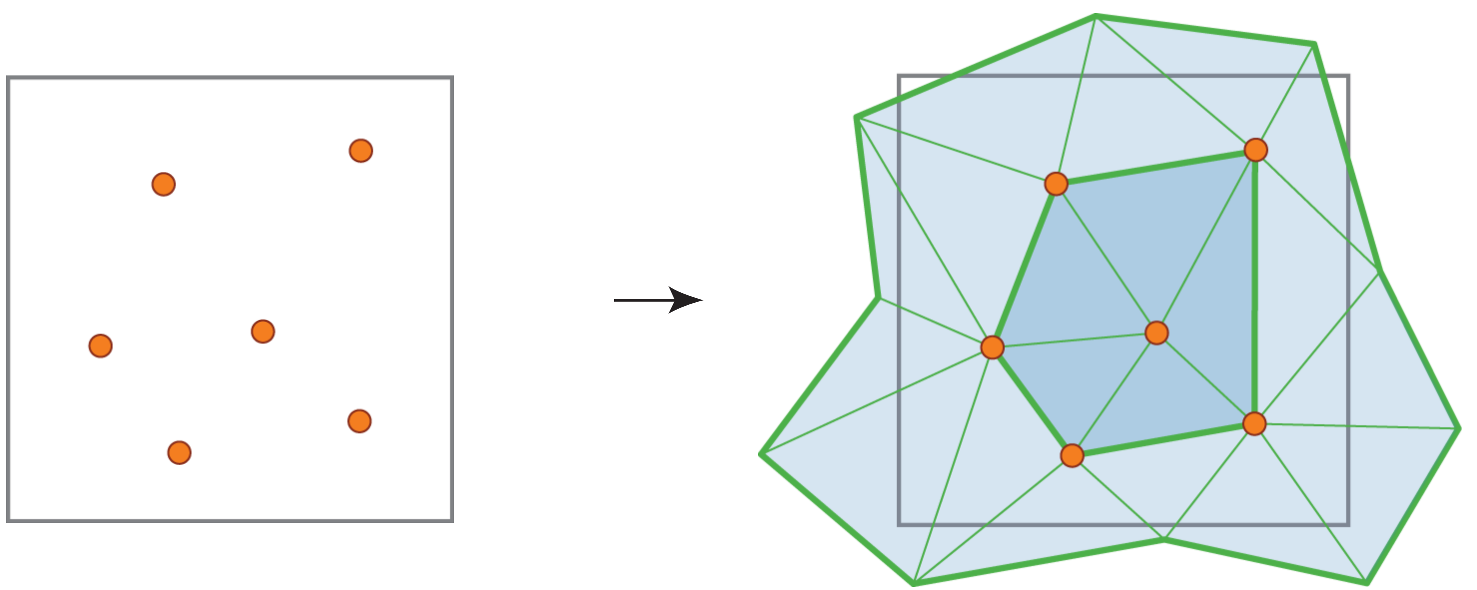


$$v_i = \{x,\ y,\ z,\ i_{t0}\}$$

$$t_i = \left\{ \begin{matrix} i_{v0}, & i_{v1}, & i_{v2}, & i_{v3} \\ i_{t0}, & i_{t1}, & i_{t2}, & i_{t3} \end{matrix} \right\}$$

PR-star Octree

- "Topology through space"
  - A spatial data structure for querying topological connectivity
- Augment PR octree with the set of tetrahedra from the mesh that are incident in its vertices
  - i.e. the tetrahedra in the star of its vertices



Generation of PR-star

Three steps

- Input is soup of tetrahedra defining a tetrahedral mesh Σ

Step 1: Vertices

- Create a PR octree ***N*** on vertices ***V*** of mesh
- Based on user selected bucket threshold k_v

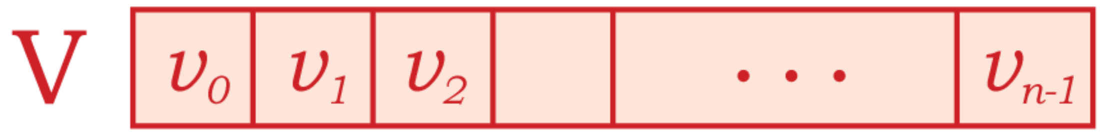
Step 2: Tetrahedra

- Add tetrahedra ***T*** to appropriate leaf nodes of ***N***

Step 3: Spatial sort

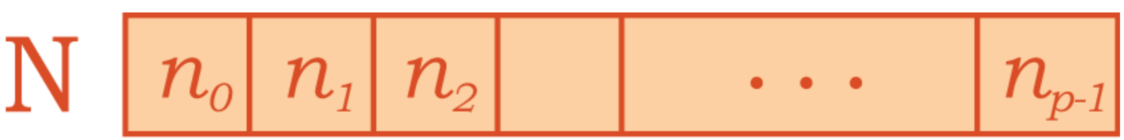
- Reorganize ***V*** and ***T*** based on spatial sorting induced by ***N***
  - Each node in ***N*** indexes a contiguous range of vertices in ***V***
  - Can be encoded via two indices v_{start} and v_{end}
- For ***T*** we store a pointer to a list of tetrahedra indices

PR-star Octree
Representation



Encodes: geometry of the mesh *[3 pointers]*

Encodes: four indices in ***V*** of its vertices *[4 pointers]*



Encodes: hierarchical octree information *[3 pointers]*

range of vertices (v_{start}, v_{end}) *[2 pointers]*

pointer to list of incident tetrahedra *[2 pointers]*

PR-star Octree

Representation

V | v_0 | v_1 | v_2 | | . . . | v_{n-1} |

Encodes: geom... [3 pointers]

T | t_0 | t_1 | t_2 | ...

Encodes: fou... ...ointers]

N | n_0 | n_1 | n_2 | ...

Encodes: hierarch... ...pointers]

range of ve... (v_{start}, v_{end}) [2 pointers]

pointer to list of incident tetrahedra [2 pointers]

Lists of tetrahedra:

Each tetrahedron appears in

- At least one octree node
- At most four octree nodes

χ – Average number of lists in which a tetrahedron appears, where

$$1 \leq \chi \leq 4$$

Evaluation

- Indexed Tetrahedral Mesh Representation
  - Fixed cost of both data structures
  - Total $4|\boldsymbol{T}| + 3|\boldsymbol{V}| \sim 27|\boldsymbol{V}|$

- IA data structure (extended)
  - Topological: $4\,|\boldsymbol{T}| + 3|\boldsymbol{V}| \sim 25|\boldsymbol{V}|$
  - Total: $8\,|\boldsymbol{T}| + 4|\boldsymbol{V}| \sim 52|\boldsymbol{V}|$

- PR-star data structure
  - Topological: $\chi\,|\boldsymbol{T}| + 7|\boldsymbol{N}| \sim 13|\boldsymbol{V}|$
  - Total: $8\,|\boldsymbol{T}| + 4|\boldsymbol{V}| \sim 40|\boldsymbol{V}|$

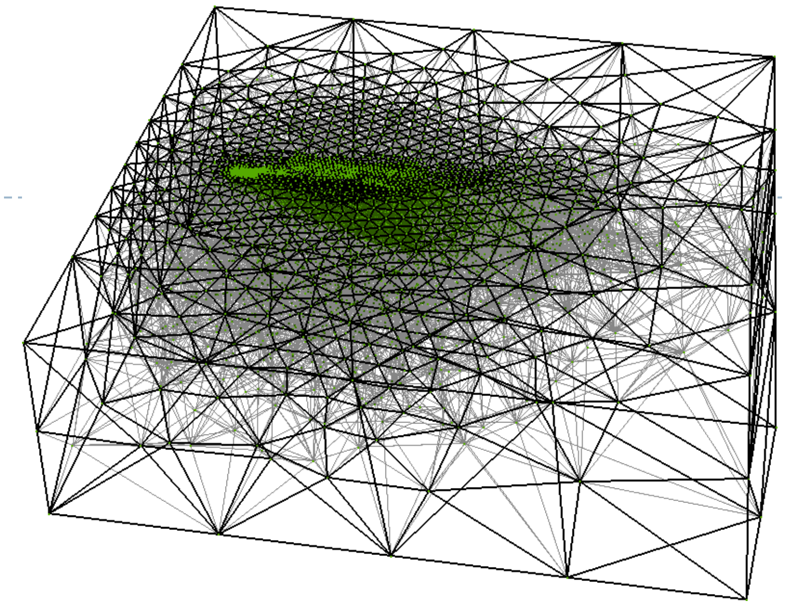
Comparison
~50% topological
~80% total storage

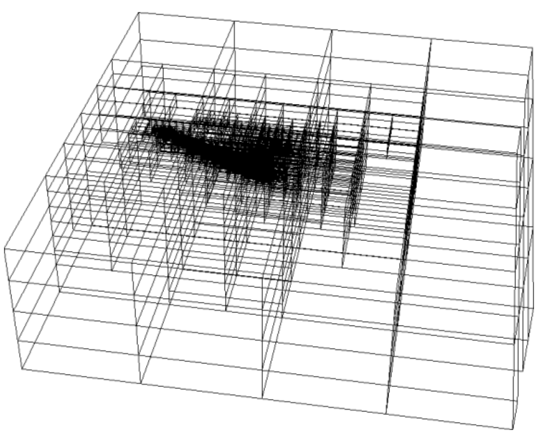
Simplifying assumptions: (see paper for details)

$|\boldsymbol{T}| \sim 6\,|\boldsymbol{V}|$ $|\boldsymbol{N}| \sim |\boldsymbol{V}| / k_v$ $\chi \sim 2$ $k_v \geq 7$

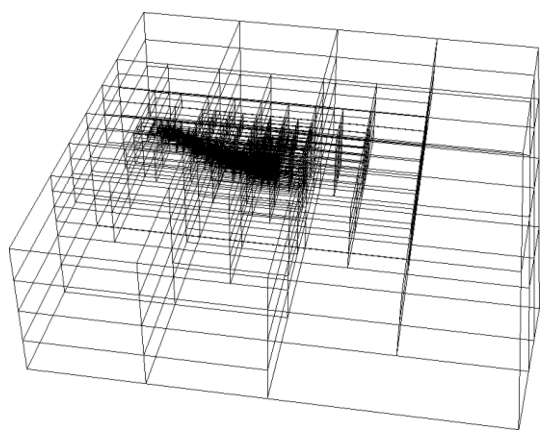
PR-star Octree:
Example

- F117 tetrahedral mesh
  - $|\boldsymbol{V}| = 48.5\ K$
  - $|\boldsymbol{T}| = 240\ K$
  - IA storage: (20.8; 43.6)

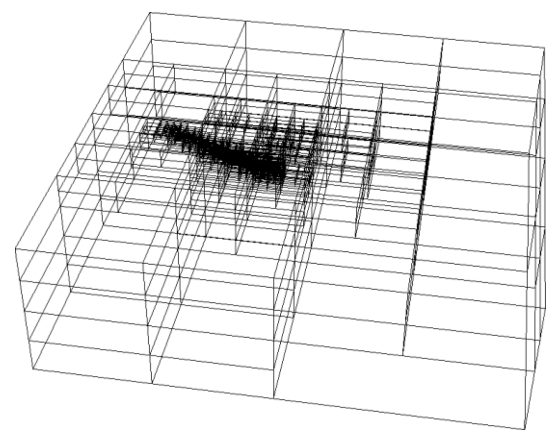




$k_v = 50$
$\chi = 2.6$; $|\boldsymbol{N}| = 4$ K
Storage: (12.8; 35.6)



$k_v = 100$
$\chi = 2.2$; $|\boldsymbol{N}| = 1.9$ K
Storage: (10.9; 33.7)



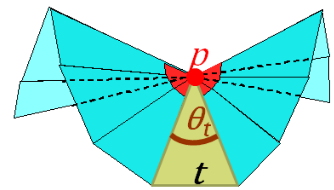
$k_v = 200$
$\chi = 2.0$; $|\boldsymbol{N}| = 1.4$ K
Storage: (10.0 ; 32.8)

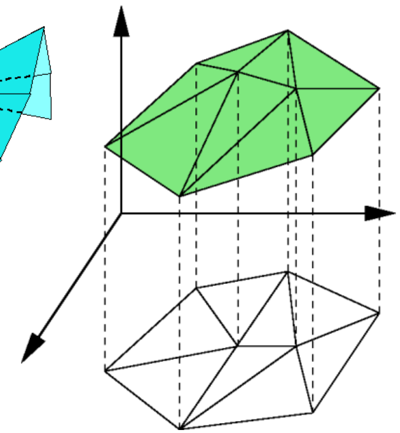
Applications of PR-star
General Strategy

- Streaming algorithm
  - Iterate through octree nodes
- For each leaf octree node
  - Step 1: Build application-dependent local data structure
  - Step 2: Process mesh locally
  - Step 3: Discard local data structure
- Cost of building data structures is amortized over multiple local operations

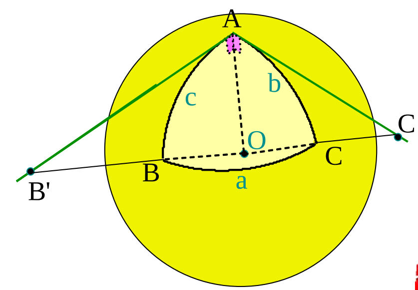
Local discrete curvature estimates

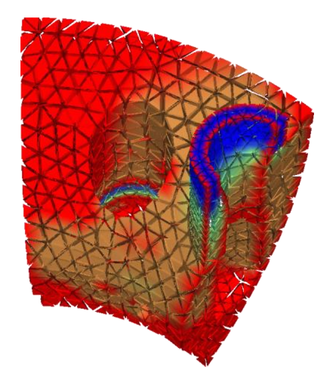
- For terrain
  - Elevations at samples in 2D domain provide embedding as 3D TIN
  - Curvature is concentrated in vertices
  - Depends on geometry of its star
    - e.g. angle deficit between 2D and 3D [Aleksandrov:1957]





- For volume data
  - Scalar values at samples in 3D domain provide embedding as 4D hypersurface
  - Curvature is concentrated in vertices
  - Depends on geometry of its star
    - e.g. angle deficit between 3D and 4D [Mesmoudi et al.:2008]





Results

Timings for generating VT and distortion

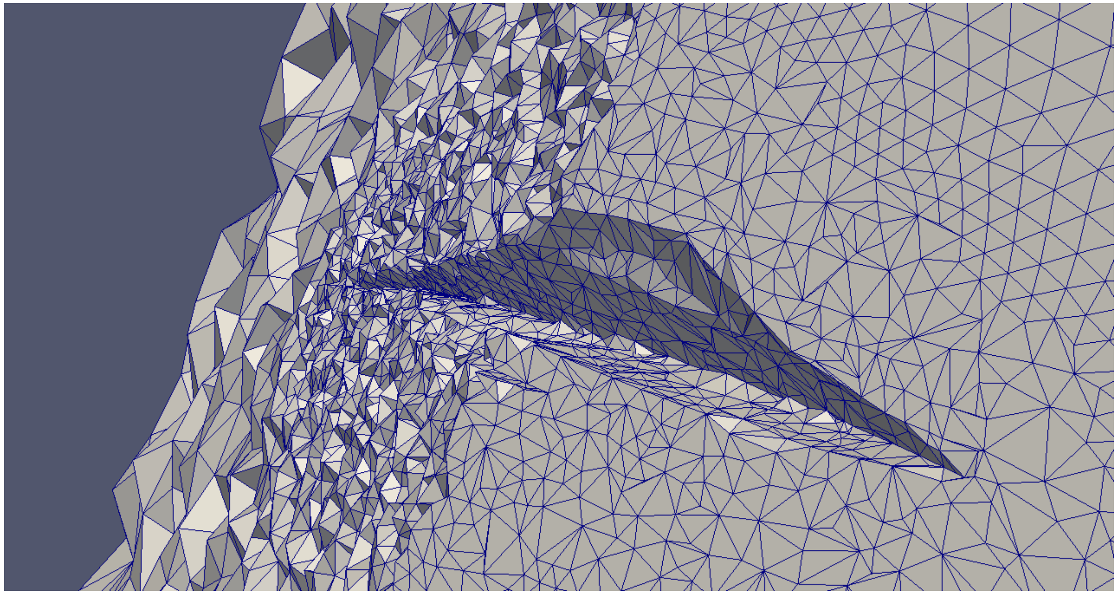
- Compared to IA data structure

- Key observations
  - Building VT is always faster for PR-star
    - Amortized cost over entire mesh
  - For small meshes with small k_v
    - Distortion computation is faster with IA
    - Value of χ plays a dominant role here
  - As mesh size increases, and as k_v increases
    - Distortion is faster with PR-star

- Trend: Effectiveness of PR-star increases with mesh size

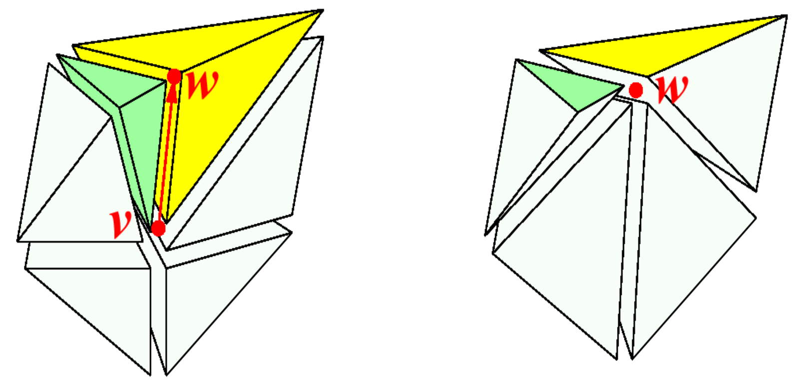
Application
Mesh simplification

- Many mesh generation processes oversample the field
- Simplification algorithms are critical to downstream processing but are resource intensive
  - Local mesh modifications require neighborhoods of vertices
  - Better results are obtained by ordering the simplifications



Local simplification
Half-edge collapse

- Simplify edge e: (w, v)
- Requires:
  - VT relation for vertex v
  - VT relation for vertex w
  - ET relation for edge e



- Steps:
  1. Delete tetrahedra in ET – applies to ***T***
  2. Modify vertices of tetrahedra in VT(v) – applies to ***V***
  3. Delete vertex v – applies to ***V***
  4. Add tetrahedra in VT(v) to VT(w) and remove ET(e)

     applies to local data structure
  5. Remove VT(v) – applies to local data structure

Simplification Algorithm

- Repeat the following until there is not change

- ALGORITHM: SIMPLIFYMESH()
  - for each node n of N
    - Generate VT relation of all vertices v_n
    - Enqueue all edges to be checked for collapse
    - while (queue is not empty)
      - Edge e = top element of queue
      - if (e passes test for simplification)
        - EDGECOLLAPSE (e)
  - SIMPLIFYOCTREE(N)   // by merging sibling leaf nodes

Results

- Compare PR-star with different k_v values
- Special case: $k_v = \infty$
  - Octree only has a single node

- Summary:
  - Similar simplification results
    - Around the same number of tetrahedra removed
    - In around the same amount of time ($\pm$ 20%)
    - using < 1% of the memory

Trend: Better results for larger meshes and larger values of k_v

Discussion

- Introduced PR-star Octree for tetrahedral meshes
  - *Spatio-Topological* approach
- Spatial index "for free"
  - One of the difficulties in topological data structures on spatial data is finding the initial vertices
- Simple global data structure
- Optimal local data structures
  - Not forced to decide in advance which operations (e.g. incidence, adjacency) to optimize
  - Efficiently build the data structure at runtime without worrying (too much) about memory consumption
- Results improve with increased mesh resolution

Limitations

- Only works for spatial meshes
  - Use traditional topological data structure for abstract complexes
- Does not replace spatial data structures
  - Not optimized for general spatial queries
    - E.g. point location (find tetrahedron containing a point)
  - Use PM-family of meshes here
  - But can handle range queries

Future work

- Tuning for parameter k_v
  - Preliminary results: k_v ~ 600-800 appears to be the sweet spot
    - Significantly smaller octrees
    - More time to build the local data structures but less time to traverse the octree
    - Not "too much" extra time to generate the local data structure
- Cache-based algorithms for non-local processing of mesh
  - e.g. simplification of edges spanning two octree nodes
  - Use a cache of expanded nodes
    - Preliminary results: Around 2% of nodes is sufficient for best results
- Exploit inherent parallelism of data structure

Thank you

- Questions? Comments?

- Anonymous reviewers
- Funding Sources
  - NSF Grant IIS-1116747
  - Italian MUIR-PRIN 2009
- Paola Magillo
- Mesh sources
  - AIM@Shape, Volvis
  - C. Silva, R. Haimes